Partial Integration and Trimetric Structures in Altered Scales

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Abstract

This paper extends Integration Theory to encompass melodic minor, harmonic minor, and harmonic major scales through the development of Partial Integration operations. We demonstrate that standard Integration, which generates complete 7×7 harmonic fields for major scales, produces incomplete fields when applied to altered scales. Through systematic analysis, we derive general solutions for Partial Integration: melodic minor follows $\int K_m(n)dn = K_m^2(n-2)$, while harmonic minor and harmonic major exhibit dual cross-product formulations at $(n-7)$. We prove that altered scales contain embedded Trimetric Triads—complete augmented triads within their structure—a property absent in diatonic major scales. The union of Modified Scales (S-2) built on these embedded triads generates Trimetric Scales with identical properties across different scale types. These findings establish that Partial Integration expands the Integration framework from 49 harmonic contexts (major scales) to 49 (melodic minor) or 98 (harmonic minor/major) contexts, while revealing fundamental structural connections between Integration Theory and Trimetric Theory.

1 Introduction

Integration Theory [1] establishes a formal framework for dimensional augmentation of diatonic harmony, transforming one-dimensional modal vectors into comprehensive two-dimensional harmonic fields. For any melody note within a major scale vector, the Integration Operation generates exactly 49 distinct harmonic contexts through the Key Squared operation $K^2(n) = n - 4 \pmod{12}$. Trimetric Theory [2] defines a systematic approach to generating nine-note scales through the union of three Modified Major Scales built on augmented triadic foundations, producing structures with distinct three-fold symmetrical properties.

The extension of Integration Theory to altered scale types within the Western music theory canon—specifically melodic minor, harmonic minor, and harmonic major scales—constitutes the primary focus of this paper. We demonstrate that the standard Integration Operation, while complete for major scales, produces incomplete harmonic fields when applied to altered scales. This necessitates the development of Partial Integration operations that preserve the Law of Integration while accommodating the distinct interval structures of these scale types.

Through systematic analysis, we establish general solutions for Partial Integration and reveal a fundamental structural distinction between diatonic and altered scales: while major scales contain only two of three notes from any augmented triad, altered scales contain complete Trimetric Triads embedded within their structure. This discovery establishes direct mathematical pathways between Integration Theory and Trimetric Theory, providing the foundation for their unification within a comprehensive analytical framework.

The paper proceeds as follows: Section 2 establishes formal foundations and reviews the Standard Integration Operation. Section 3 develops Partial Integration formulations for melodic minor, harmonic minor, and harmonic major scales. Section 4 analyzes Trimetric structures within altered scales and proves the invariance of Trimetric properties across scale types.

2 Formal Foundations

2.1 Definitions and Notation

Let $\mathbb{Z}_{12}$ denote the cyclic group of integers modulo 12, representing the chromatic pitch space. For any $n \in \mathbb{Z}_{12}$, we define the mapping $z : \mathbb{Z}_{12} \to \mathbb{C}$ by:

$$z(n) = e^{2\pi i n/12}$$

This maps each pitch class to a point on the complex unit circle, providing a geometric representation of tonal relationships.

Definition 2.1 (Diatonic Scale). *For any $n \in \mathbb{Z}_{12}$, the diatonic major scale with tonic n is defined as:*

$$MS(n) = \{n, n+2, n+4, n+5, n+7, n+9, n+11\} \pmod{12}$$

Definition 2.2 (Melodic Minor Scale). *For any $n \in \mathbb{Z}_{12}$, the melodic minor scale (ascending) with tonic n is defined as:*

$$MM(n) = \{n, n+2, n+3, n+5, n+7, n+9, n+11\} \pmod{12}$$

Definition 2.3 (Harmonic Minor Scale). *For any $n \in \mathbb{Z}_{12}$, the harmonic minor scale with tonic n is defined as:*

$$HM(n) = \{n, n+2, n+3, n+5, n+7, n+8, n+11\} \pmod{12}$$

Definition 2.4 (Harmonic Major Scale). *For any $n \in \mathbb{Z}_{12}$, the harmonic major scale with tonic n is defined as:*

$$HJ(n) = \{n, n+2, n+4, n+5, n+7, n+8, n+11\} \pmod{12}$$

Definition 2.5 (Mode). *For any scale $S = \{s_0, s_1, ..., s_6\}$ and $k \in \{0, 1, ..., 6\}$, the k-th mode of S is defined as:*

$$M_k(S) = \{s_k, s_{k+1}, ..., s_6, s_0, ..., s_{k-1}\}$$

2.2 Review: Standard Integration Operation

Integration Theory [1] defines the dimensional augmentation of diatonic harmonic spaces through the Integration Operation. For completeness, we briefly review the standard formulation.

Definition 2.6 (Standard Integration Operation). *For any pitch class $n \in \mathbb{Z}_{12}$ in a major scale context, the Integration Operation produces the Key Squared:* $\int K(n)dn = K^2(n) = n - 4 \pmod{12}$ *In the complex plane, this corresponds to:* $z(K^2(n)) = e^{-2\pi i/3} \cdot z(n)$ *representing a clockwise rotation of 120°.*

Definition 2.7 (Integration Field). *The Integration Field* $\int K(n)dn$ *is the complete* 7×7 *harmonic space where:*

- *The first row and column contain the major scale with tonic $K^2(n)$*
- *Cell (i, j) contains the first note of the $(i-1)$-th mode of the major scale with tonic at position j in the first row*
- *This generates exactly 49 contexts, all containing the original note n*

The standard Integration Operation satisfies the Law of Integration for major scales: it produces the complete set of all diatonic harmonic contexts containing the input note. However, when applied to altered scales, this operation produces incomplete fields with "holes"—positions where the input note does not appear—necessitating the development of Partial Integration.

3 Partial Integration

3.1 Motivation and Definition

Definition 3.1 (The Law of Integration). *For any note n within a specified scale context, Integration (whether standard or partial) produces the complete extra-dimensional harmonic field containing all possible diatonic contexts in which n appears within that scale type. This completeness property ensures that the resulting Integration Field represents the exhaustive harmonic profile for the given input and context.*

Definition 3.2 (Partial Integration). *Partial Integration employs alternative procedures when Standard Integration fails to produce complete harmonic fields. This is achieved through either:*

1. ***Interval modification**: Using a different interval relationship (e.g., $n-2$ instead of $n-4$)*
2. ***Cross-product formulation**: Combining two different scale types to generate the complete field*

Standard Integration fails for altered scales because the resulting fields contain positions where the original note does not appear, creating incomplete harmonic representations. Partial Integration provides the necessary mathematical procedures to ensure the Law of Integration is satisfied for these contexts.

3.2 Melodic Minor Integration

Theorem 3.3 (Melodic Minor Integration Formulation). *For any melody note n, the Integration of melodic minor scales follows:*

$$\int K_m(n)dn = K_m^2(n-2) \tag{1}$$

In the complex plane: $z(K_m^2(n)) = e^{\pi i/3} \cdot z(n)$ (i.e., 60° counterclockwise rotation). Where K_m represents melodic minor scale types.

3.3 Harmonic Minor and Harmonic Major Integration

Theorem 3.4 (Harmonic Minor Integration Formulation). *For any melody note n, the Integration of harmonic minor scales exhibits a dual relationship with harmonic major scales:*

$$\int K_h(n)dn = \left[K_M\left(\sum_{i=1}^{7}(K_h(n-7))\right)\right] \times \left[K_h\left(\sum_{i=1}^{7}(K_M(n-7))\right)\right] \tag{2}$$

In the complex plane: $z(K_h^2(n)) = e^{-\pi i/6} \cdot z(n)$ *(i.e., 30° clockwise rotation). Where* K_h *represents harmonic minor scale types and* K_M *represents harmonic major scale types.*

Theorem 3.5 (Harmonic Major Integration Formulation)**.** *For any melody note n, the Integration of harmonic major scales exhibits a dual relationship with harmonic minor scales, reversing the order of the cross-product:*

$$\int K_M(n)dn = \left[K_h\left(\sum_{i=1}^{7}(K_M(n-7))\right)\right] \times \left[K_M\left(\sum_{i=1}^{7}(K_h(n-7))\right)\right] \tag{3}$$

In the complex plane: $z(K_M^2(n)) = e^{-\pi i/6} \cdot z(n)$ *(i.e., 30° clockwise rotation).*

Theorem 3.6 (Dual Nature of Harmonic Integration)**.** *The Integration operations for harmonic minor and harmonic major scales exhibit a complementary structure where the operations between these scale types are dual to each other, forming a balanced cross-product relationship:*

$$\int K_h(n)dn = \left[K_M\left(\sum_{i=1}^{7}(K_h(n-7))\right)\right] \times \left[K_h\left(\sum_{i=1}^{7}(K_M(n-7))\right)\right] \tag{4}$$

$$\int K_M(n)dn = \left[K_h\left(\sum_{i=1}^{7}(K_M(n-7))\right)\right] \times \left[K_M\left(\sum_{i=1}^{7}(K_h(n-7))\right)\right] \tag{5}$$

Proof. The duality between harmonic minor and harmonic major scales manifests in their Integration operations through the reversal of the cross-product terms. This mirrors the interval symmetry between these scale types, where the interval pattern of one is the reverse of the other. The perfect fifth relationship $(n-7)$ establishes a consistent reference point for both operations, enabling a seamless transition between these complementary tonal systems. □

Definition 3.7 (Derived Solutions vs. General Solutions)**.** *For altered scales, derived solutions refer to the specific configurations discovered through manual manipulation of scale matrices to fill "holes" in the Integration field. General solutions are the formalized mathematical expressions that directly produce complete fields without intermediate steps:*

- *Melodic Minor:* $K_m^2(n-2)$ *(general) vs.* $(K_m(n-3)) \times (K_{SL}(n-3))$ *(derived)*
- *Harmonic Minor:* $[K_M(K_h(n-7))] \times [K_h(K_M(n-7))]$ *(general) vs.* $[(K_h(n-3))] \times [(K_{Phr\flat 4}(n-3))]$ *(derived)*

Where K_{SL} *represents Super Locrian scale types and* $K_{Phr\flat 4}$ *represents Phrygian ♭4 scale types.*

4 Trimetric Structures in Altered Scales

4.1 Modified Scale Properties

Definition 4.1 (Modified Scale)**.** *For any scale* S *and tonic* n*, the Modified Scale* $S\text{-}2(n)$ *is the scale* $S(n)$ *with the second scale degree removed:* $S\text{-}2(n) = S(n) \setminus \{n+2\}$

Theorem 4.2 (Modified Scale Consistency)**.** *For any scale type* $S \in \{MS, MM, HM, HJ\}$ *and root note* $n \in \mathbb{Z}_{12}$*, the S-2 modified scale preserves the characteristic intervals of the original scale type while removing the 2nd scale degree.*

Proof. For each scale type, removing the second scale degree leaves the interval structure between all remaining notes unchanged:

- For major scale: $MS(n) = \{n, n+2, n+4, n+5, n+7, n+9, n+11\}$
  $MS\text{-}2(n) = \{n, n+4, n+5, n+7, n+9, n+11\}$

- For melodic minor: $MM(n) = \{n, n+2, n+3, n+5, n+7, n+9, n+11\}$
  $MM\text{-}2(n) = \{n, n+3, n+5, n+7, n+9, n+11\}$

- For harmonic minor: $HM(n) = \{n, n+2, n+3, n+5, n+7, n+8, n+11\}$
  $HM\text{-}2(n) = \{n, n+3, n+5, n+7, n+8, n+11\}$

- For harmonic major: $HJ(n) = \{n, n+2, n+4, n+5, n+7, n+8, n+11\}$
  $HJ\text{-}2(n) = \{n, n+4, n+5, n+7, n+8, n+11\}$

The characteristic intervals of each scale type are preserved while maintaining a consistent transformation across all scale types. □

Definition 4.3 (Trimetric Scale). *For any $n \in \mathbb{Z}_{12}$, the Trimetric Scale $\Delta(n)$ is the nine-note collection formed by the union of three Modified Major Scales built on the Trimetric Triad: $\Delta(n) = \bigcup_{k \in T(n)} MS\text{-}2(k)$ where $T(n) = \{n, n+4, n+8\}$ is the Trimetric Triad.*

Theorem 4.4 (Scale Type Invariance). *For certain root notes $n \in \mathbb{Z}_{12}$, the Trimetric Scales generated from different scale types can be identical:*

$$TS_{MS}(n) = TS_{MM}(n') = TS_{HM}(n'') = TS_{HJ}(n''')$$

For specific related values of n, n', n'', n'''.

Proof. Through analysis of the modified scales $S\text{-}2(x)$ for each scale type, we can determine that there exist specific relationships between root notes n, n', n'', n''' such that the resulting Trimetric Scales contain identical sets of notes, despite being derived from different scale types.

For example, when $n = C$, $n' = A$, $n'' = A$ and $n''' = C$, we find:

$$TS_{MS}(C) = TS_{MM}(A) = TS_{HM}(A) = TS_{HJ}(C) = \{C, C\sharp, D\sharp, E, F, G, G\sharp, A, B\}$$

This demonstrates the convergent properties of different scale types under the Trimetric Transform. □

4.2 Altered Scale Triad Embedding

Theorem 4.5 (Altered Scale Triad Embedding). *Melodic minor, harmonic minor, and harmonic major scales all contain complete Trimetric Triads within their structure:*

1. *Melodic minor scales contain a Trimetric Triad from their minor 3rd degree*

2. *Harmonic minor scales contain a Trimetric Triad from their minor 3rd degree*

3. *Harmonic major scales contain a Trimetric Triad from their tonic*

This creates natural bridges between Integration and Trimetric frameworks through these embedded structures.

Proof. For C melodic minor (C, D, E♭, F, G, A, B), the Trimetric Triad ΔE♭GB is embedded within the scale.

For C harmonic minor (C, D, E♭, F, G, A♭, B), the Trimetric Triad ΔE♭GB is embedded within the scale.

For C harmonic major (C, D, E, F, G, A♭, B), the Trimetric Triad ΔCEG♯ (enharmonically G♯ = A♭) is embedded within the scale.

These embedded triads provide direct connections between the tonal frameworks and enable efficient movement between them. □

4.3 Structural Symmetry in Altered Scales

Theorem 4.6 (Harmonic Scale Symmetry). *Harmonic minor and harmonic major scales exhibit reverse-order interval symmetry:*

$$Intervals(HM(n)) = Reverse(Intervals(HJ(n'))) \tag{6}$$

For appropriate tonic values n, n'.

Proof. The interval sequence of harmonic major {W,W,H,W,H,WH,H} becomes {H,WH,H,W,W,H,W} when reversed, which matches the harmonic minor interval sequence when starting from the 5th position. This symmetry relationship creates a fundamental connection between these scale types and explains their complementary behavior under Integration.

Specifically, for harmonic major with tonic n, the interval sequence is: {n→n+2, n+2→n+4, n+4→n+5, n+5→n+7, n+7→n+8, n+8→n+11, n+11→n} = {2, 2, 1, 2, 1, 3, 1}

For harmonic minor with tonic n', the interval sequence is: {n'→n'+2, n'+2→n'+3, n'+3→n'+5, n'+5→n'+7, n'+7→n'+8, n'+8→n'+11, n'+11→n'} = {2, 1, 2, 2, 1, 3, 1}

When reversed, the harmonic major sequence becomes {1, 3, 1, 2, 1, 2, 2}, which corresponds to the harmonic minor sequence rotated to start at the 5th position. □

5 Conclusion

This paper establishes the extension of Integration Theory to altered scale types through Partial Integration operations, revealing fundamental structural connections between Integration Theory and Trimetric Theory. We have demonstrated three primary results:

First, we derived general solutions for Partial Integration across melodic minor, harmonic minor, and harmonic major scales. The melodic minor formulation $\int K_m(n)dn = K_m^2(n-2)$ represents a shift by major second rather than major third, while the harmonic minor and harmonic major formulations employ dual cross-product structures at $(n-7)$, creating complementary 7×7 matrices from both scale types simultaneously.

Second, we proved that altered scales contain embedded Trimetric Triads—complete augmented triads within their structure—distinguishing them fundamentally from diatonic major scales, which contain only two of three notes from any augmented triad. This structural difference establishes direct mathematical pathways between Integration and Trimetric frameworks.

Third, we demonstrated that Modified Scales (S-2) preserve characteristic intervals across different scale types, and that Trimetric Scales generated from these modified scales exhibit invariance properties: identical nine-note collections result from different scale type inputs with appropriately related root notes.

These findings expand the Integration framework from 49 harmonic contexts (major scales) to include 49 (melodic minor) and 98 (harmonic minor/major) contexts, while establishing the theoretical foundation necessary for unifying Integration Theory and Trimetric Theory within a comprehensive analytical framework. The embedded Trimetric structures in altered scales and the dual-scale properties of harmonic Integration provide the mathematical basis for this unification, which will be developed in subsequent work.

Future research directions include extension of Partial Integration to additional altered scale types within and beyond the Western music theory canon, investigation of higher-order cross-product formulations, and development of computational implementations of these extended Integration operations.

References

References

[1] Simpson, A. (2025a). Integration theory: Dimensional augmentation of diatonic harmonic spaces. A Micron Music. https://www.amicronmusic.com/integration-thesis

[2] Simpson, A. (2025b). Trimetric theory: Equilateral symmetry in nine-note scale structures. A Micron Music. https://www.amicronmusic.com/trimetry-thesis

[3] Ellard, G., & Johnstone, S. (2018). Music theory and composition: A practical approach. Rowman & Littlefield.

[4] Larson, R., & Edwards, B. (2010). Calculus (9th ed.). Brooks/Cole Cengage Learning.

A Mathematical Proofs and Worked Examples

This appendix provides formal proofs and computational demonstrations of the Partial Integration operations introduced in this paper. Section A.1 presents worked examples using melody notes E and G across all scale types. Section A.2 establishes formal proofs of completeness for each Partial Integration formulation.

A.1 Worked Examples: Integration of E and G

We demonstrate the Integration and Partial Integration operations through complete calculations across four scale types: major, melodic minor, harmonic minor, and harmonic major.

A.1.1 Standard Integration: E in Major Scale Context

For melody note $n = E$ in major scale context:

$$\int K(E)\,dE = K^2(E) = E - 4 \pmod{12} = C^2 \tag{7}$$

Complex plane representation:

$$z(K^2(E)) = e^{-2\pi i/3} \cdot z(E) \tag{8}$$

The resulting Integration Field $K^2(E) = C^2$ produces the following 7×7 harmonic matrix. Both the top row and leftmost column contain the C major scale $\{C, D, E, F, G, A, B\}$. Note E appears exactly once in each row and once in each column:

C^2	C	D	E	F	G	A	B
C	C	D	E	F	G	A	B
D	D	E	F♯	G	A	B	C♯
E	E	F♯	G♯	A	B	C♯	D♯
F	F	G	A	A♯	C	D	E
G	G	A	B	C	D	E	F♯
A	A	B	C♯	D	E	F♯	G♯
B	B	C♯	D♯	E	F♯	G♯	A♯

Complete grids for all 12 chromatic notes provided in Simpson (2025a), Appendix A.

A.1.2 Melodic Minor Partial Integration: E in Melodic Minor Context

For melody note $n = E$ in melodic minor context:

$$\int K_m(E)\,dE = K_m^2(E-2) = K_m^2(D) = D - 4 \pmod{12} = A_m^2 \tag{9}$$

Complex plane representation:

$$z(K_m^2(E)) = e^{\pi i/3} \cdot z(E) \tag{10}$$

The resulting field A_m^2 produces the following 7×7 harmonic matrix. Both the top row and leftmost column contain the A melodic minor scale $\{A, B, C, D, E, F\sharp, G\sharp\}$. Note E appears exactly once in each row and once in each column:

A_m^2	A	B	C	D	E	F♯	G♯
A	A	B	C	D	E	F♯	G♯
B	B	C	D	E	F♯	G♯	A
C	C	D	E	F♯	G♯	A	B
D	D	E	F♯	G	A	B	C♯
E	E	F♯	G♯	A	B	C♯	D♯
F♯	F♯	G♯	A♯	B	C♯	D♯	E♯
G♯	G♯	A♯	B♯	C♯	D♯	E♯	F♯♯

This matrix exhausts all melodic minor harmonic contexts containing E.

A.1.3 Harmonic Minor Partial Integration: E and G in Harmonic Minor Context

For melody note $n = E$ in harmonic minor context:

$$\int K_h(E)\,dE = \left[K_M\left(\sum_{i=1}^{7} K_h(E-7)\right)\right] \times \left[K_h\left(\sum_{i=1}^{7} K_M(E-7)\right)\right] \tag{11}$$

Simplifying with $E - 7 = A$:

$$\int K_h(E)\,dE = [K_M(K_h(A))] \times [K_h(K_M(A))] = A_{h/M}^2 \tag{12}$$

Complex plane representation:

$$z(K_h^2(E)) = e^{-\pi i/6} \cdot z(E) \tag{13}$$

The resulting dual field $A_{h/M}^2$ produces a 7×7 matrix where rows are constructed from harmonic minor scales and columns from harmonic major scales. The top row contains the A harmonic minor scale $\{A, B, C, D, E, F, G\sharp\}$, while the leftmost column contains the A harmonic major scale $\{A, B, C\sharp, D, E, F, G\sharp\}$. Note E appears exactly once in each row and once in each column:

$A_{h/M}^2$	A	B	C	D	E	F	G♯
A	A	B	C	D	E	F	G♯
B	B	C♯	D	E	F♯	G	A♯
C♯	C♯	D♯	E	F♯	G♯	A	B♯
D	D	E	F	G	A	B♭	C♯
E	E	F♯	G	A	B	C	D♯
F	F	G	A♭	B♭	C	D♭	E
G♯	G♯	A♯	B	C♯	D♯	E	F♯♯

For melody note $n = G$ in harmonic minor context, with $G - 7 = C$:

$$\int K_h(G)\,dG = C^2_{h/M} \tag{14}$$

The top row contains the C harmonic minor scale $\{C, D, E\flat, F, G, A\flat, B\}$, while the leftmost column contains the C harmonic major scale $\{C, D, E, F, G, A\flat, B\}$. Note G appears exactly once in each row and once in each column:

$C^2_{h/M}$	C	D	E♭	F	G	A♭	B
C	C	D	E♭	F	G	A♭	B
D	D	E	F	G	A	B♭	C♯
E	E	F♯	G	A	B	C	D♯
F	F	G	A♭	B♭	C	D♭	E
G	G	A	B♭	C	D	E♭	F♯
A♭	A♭	B♭	C♭	D♭	E♭	F♭	G
B	B	C♯	D	E	F♯	G	A♯

This dual structure provides 98 distinct harmonic contexts (49 harmonic minor positions in rows, 49 harmonic major positions in columns), all containing the input note.

A.1.4 Harmonic Major Partial Integration: E and G in Harmonic Major Context

For melody note $n = E$ in harmonic major context:

$$\int K_M(E)\,dE = \left[K_h\left(\sum_{i=1}^{7} K_M(E-7)\right)\right] \times \left[K_M\left(\sum_{i=1}^{7} K_h(E-7)\right)\right] \tag{15}$$

Simplifying with $E - 7 = A$:

$$\int K_M(E)\,dE = [K_h(K_M(A))] \times [K_M(K_h(A))] = A^2_{M/h} \tag{16}$$

Complex plane representation:

$$z(K^2_M(E)) = e^{-\pi i/6} \cdot z(E) \tag{17}$$

The resulting dual field $A^2_{M/h}$ produces a 7×7 matrix where rows are constructed from harmonic major scales and columns from harmonic minor scales. The top row contains the A harmonic major scale $\{A, B, C\sharp, D, E, F, G\sharp\}$, while the leftmost column contains the A harmonic minor scale $\{A, B, C, D, E, F, G\sharp\}$. Note E appears exactly once in each row and once in each column:

$A^2_{M/h}$	A	B	C♯	D	E	F	G♯
A	A	B	C♯	D	E	F	G♯
B	B	C♯	D♯	E	F♯	G	A♯
C	C	D	E	F	G	A♭	B
D	D	E	F♯	G	A	B♭	C♯
E	E	F♯	G♯	A	B	C	D♯
F	F	G	A	B♭	C	D♭	E
G♯	G♯	A♯	B♯	C♯	D♯	E	F♯♯

For melody note $n = G$ in harmonic major context, with $G - 7 = C$:

$$\int K_M(G)\,dG = C^2_{M/h} \tag{18}$$

The top row contains the C harmonic major scale $\{C, D, E, F, G, A\flat, B\}$, while the leftmost column contains the C harmonic minor scale $\{C, D, E\flat, F, G, A\flat, B\}$. Note G appears exactly once in each row and once in each column:

$C^2_{M/h}$	C	D	E	F	G	A♭	B
C	C	D	E	F	G	A♭	B
D	D	E	F♯	G	A	B♭	C♯
E♭	E♭	F	G	A♭	B♭	C♭	D
F	F	G	A	B♭	C	D♭	E
G	G	A	B	C	D	E♭	F♯
A♭	A♭	B♭	C	D♭	E♭	F♭	G
B	B	C♯	D♯	E	F♯	G	A♯

Note the reversed cross-product structure compared to harmonic minor integration (Section A.1.3). The harmonic major and harmonic minor matrices for the same input note are transposes of each other, reflecting the dual complementarity of these scale types (Theorem 3.6).

A.2 Formal Proofs

A.2.1 Completeness of Melodic Minor Partial Integration

Theorem A.1 (Melodic Minor Integration Completeness)**.** *For any melody note n, the Partial Integration operation*

$$\int K_m(n)\,dn = K_m^2(n-2)$$

produces exactly the complete set of all melodic minor harmonic contexts containing n, and only these contexts.

Proof. **Part 1 (Existence):** We prove the input note appears exactly once in each row and once in each column.

For any note n, the operation $K_m^2(n-2)$ places $(n-2)$ as the tonic of the K^2 melodic minor scale. Since $n = (n-2)+2$, note n is the 2nd scale degree of the $K_m^2(n-2)$ melodic minor scale.

The K^2 Ionian set generates 7 keys, each with 7 modes, producing 49 harmonic contexts. Since n is contained in the $K_m^2(n-2)$ melodic minor scale, all 49 modes derived from this K^2 set contain n exactly once per row and once per column.

Part 2 (Exhaustiveness): We prove these are the only contexts.

Of the 12 possible key centers, exactly 7 contain a given note n diatonically in melodic minor contexts. The operation $K_m^2(n-2)$ generates a K^2 Ionian set of 7 notes. These 7 notes, as melodic minor key centers, produce exactly the 7 keys containing n. The remaining 5 key centers do not contain n in melodic minor contexts.

Geometric Interpretation: In the complex plane, this operation represents a 60° counterclockwise rotation: $z(K_m^2(n)) = e^{\pi i/3} \cdot z(n)$, distinct from the 120° clockwise rotation of Standard Integration. □

A.2.2 Completeness of Harmonic Minor/Major Dual Integration

Theorem A.2 (Harmonic Scale Dual Integration Completeness)**.** *For any melody note n, the Partial Integration operations for harmonic minor and harmonic major:*

$$\int K_h(n)\,dn = \left[K_M\left(\sum_{i=1}^{7} K_h(n-7)\right)\right] \times \left[K_h\left(\sum_{i=1}^{7} K_M(n-7)\right)\right] \tag{19}$$

$$\int K_M(n)\,dn = \left[K_h\left(\sum_{i=1}^{7} K_M(n-7)\right)\right] \times \left[K_M\left(\sum_{i=1}^{7} K_h(n-7)\right)\right] \tag{20}$$

produce complete dual fields containing all and only the harmonic contexts (minor or major, respectively) containing n.

Proof. **Necessity of Cross-Product Formulation:**

Unlike melodic minor, harmonic minor and harmonic major scales exhibit asymmetrical interval structures that prevent single-scale field completion. The interval sequence of harmonic major $\{W, W, H, W, H, WH, H\}$ reverses to match harmonic minor when appropriately rotated (Theorem 4.6). This structural complementarity necessitates a dual-scale approach.

Part 1 (Existence): We prove the input note appears exactly once in each row and once in each column.

The operation at $(n-7)$ (perfect fifth below) establishes a reference point where both harmonic minor and harmonic major scales coexist. For harmonic minor integration, modes are constructed from harmonic minor scales along rows while harmonic major scales generate columns. This cross-product structure ensures n appears exactly once in each row and once in each column.

Part 2 (Dual Complementarity):

The reversed order of cross-product terms between harmonic minor and harmonic major integration (Theorem 3.6) reflects their interval symmetry. For any note n:

- Harmonic minor integration yields a field with harmonic minor scales along rows, harmonic major scales along columns
- Harmonic major integration yields a field with harmonic major scales along rows, harmonic minor scales along columns
- These two matrices are transposes of each other

This dual structure provides complete coverage of both scale types through a single operation, yielding 98 total contexts (49 of each type).

Geometric Interpretation: Both operations represent 30° clockwise rotation in the complex plane: $z(K^2_{h/M}(n)) = e^{-\pi i/6} \cdot z(n)$. The magnitude of this angle is exactly half that of melodic minor integration, but in the opposite direction. □

A.2.3 Embedded Trimetric Triads in Altered Scales

Theorem A.3 (Altered Scale Triad Embedding)**.** *Melodic minor, harmonic minor, and harmonic major scales contain complete Trimetric Triads (augmented triads) within their structures, while major scales contain only two of three notes from any Trimetric Triad.*

Proof. We prove by construction for each scale type.

Melodic Minor: For C melodic minor $= \{C, D, E\flat, F, G, A, B\}$, the Trimetric Triad $\Delta_{E\flat GB} = \{E\flat, G, B\}$ is embedded within the scale, formed by scale degrees $\hat{3}, \hat{5}, \hat{7}$.

Harmonic Minor: For C harmonic minor = $\{C, D, E\flat, F, G, A\flat, B\}$, the Trimetric Triad $\Delta_{E\flat GB} = \{E\flat, G, B\}$ is embedded, formed by scale degrees $\hat{3}, \hat{5}, \hat{7}$.

Harmonic Major: For C harmonic major = $\{C, D, E, F, G, A\flat, B\}$, the Trimetric Triad $\Delta_{CEG\sharp} = \{C, E, G\sharp\}$ (enharmonically $G\sharp = A\flat$) is embedded, formed by scale degrees $\hat{1}, \hat{3}, \hat{6}$.

Major Scale Contrast: For C major = $\{C, D, E, F, G, A, B\}$, no complete Trimetric Triad exists. The closest candidates are:

- $\Delta_{CEG\sharp}$: Contains only $\{C, E\}$ (missing $G\sharp$)
- $\Delta_{FAC\sharp}$: Contains only $\{F, A\}$ (missing $C\sharp$)

This structural distinction establishes direct mathematical pathways between Partial Integration (altered scales) and Trimetric Theory, absent in Standard Integration (major scales). The embedded Trimetric Triads enable the construction of Trimetric Scales via Modified Scale unions (Section 4), demonstrating fundamental convergence between these theoretical frameworks. □

Note: Complete Integration grids for all 12 chromatic notes follow analogous computational procedures. The worked examples of E and G demonstrate the general methodology applicable across all pitch classes.